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# Mathematics News Letter

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To mathematics in general, to the following causes in particular is this journal dedicated: (1) the common problems of grade, high school and college mathematics teaching, (2) the disciplines of mathematics, (3) the promotion of M. A. of A. and N. C. of T. of M. projects.



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## ANALYSIS IN LEARNING

As the Mathematics News Letter is intended to weld together more or less diverse view-points of grade, high school and college mathematics teachers, so, we believe it might be possible for the psychologist, the educator, the mathematician to serve each other by using some common organ of exposition—an organ sufficiently free from technique to permit its matter be fairly understandable by the lay reader. If such a medium already exists we certainly need to be informed of it.

An instance of its possible usefulness is connected with our effort to emphasize the above topic. The psychology implied in what we shall say about it may not be up to the mark. It may be unsound pedagogy to assume that the average freshman can be led by any method to a measure of interest in the game of logical deduction played with the formulas of trigonometry, or other mathematical material.

Nor should the over critical reader here remind us of the value of mere experiment to determine the facts in question. A college mathematics classroom is no experimental laboratory. Psychological and educational experiment should, normally, in our judgment, be pre-collegiate. Furthermore, it should be

so scientific in character that its results are not subject to question. Until such scientific proof to the contrary is forthcoming, we shall continue to assume, as we have done in the past, that an hour's "strain of attention" to a scheme for systematic and scientific handling of his lesson material—primary object of his college program—should not be fatal to the average eighteen-year-old youth.

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After all is said, if there is such a thing as mental discipline, the measure of the discipline must be some function of the newness of the subject matter. Once one's ability to read a language without effort has been acquired there is no longer the discipline of struggle for mental adaptation to something just out of the mental reach, and the language ceases to have educational value for one. Similarly, when the basic formulae of trigonometry have been derived by thoroughly understood methods of analysis, so that they have become familiar and unconscious thought tools, further disciplines must be sought in other and logically subsequent mathematical programs. The truth here announced—familiar to all—only emphasizes the critical importance of a program which aims as far as possible to permit the student, confronted for the first time with a wholly new branch of science, TO MAKE HIS OWN MENTAL ADAPTATIONS TO ITS MATERIAL, TO GO THROUGH AT LEAST SOME OF HIS CONSCIOUS STRUGGLES AT MASTERING IT WITH BUT LITTLE, IF ANY, AID.

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Of course there are limitations to the last-named condition. The modern program of learning is great and usually compressed within small time periods. There may have been a time when absolute self-guidance in mathematics was possible. But certainly now it has passed. The guidance that is allowable should be largely fundamental. According with this principle is the fundamental character of the suggestions which follow.

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Suppose a beginner in trigonometry in possession, after two or three lessons, of the following knowledge: (1) The product, quotient and reciprocal relations of the six trigonometric functions; (2), the formulae: $\sin^2 A+\cos^2 A=1$, $\tan^2 A+1=\sec^2 A$, $\cot^2 A+1=\csc^2 A$. Suppose that the truth of other trigonometric

identities is to be established by these principles and ordinary algebraic operations.

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(1) Show that $$\frac{\sin x + \tan x}{\cot x + \csc x} = \sin x \tan x$$

Is it better to try to transform the left function into the right or the right one into the left?

**Answer.** Apparently the former is better as, usually, to develop a fractional into an integral form is easier than to develop the integral into the fractional one.

Determine by analysis a correct first step.

**Answer.** As the right function is made up only of the sine and the tangent, evidently that step is correct which transforms the left function into one expressed only in terms of the sine and the tangent. Thus, we have

$$\frac{\sin x+\tan x}{\cot x+\csc x} = \frac{\sin x+\tan x}{1/\tan x+1/\sin x}$$

$$= \frac{\sin x+\tan x}{\dfrac{\sin x+\tan x}{\sin x \tan x}}$$

$$= \sin x \tan x$$

(2) Show that $\cot^2 x - \cos^2 x = \cot^2 x \cos^2 x$

Determine by analysis a correct first step in identifying these two functions.

**Answer:** Since the left function has the term "$\cot^2 x$," a proper step is to transform the right function so that it also shall have the term "$\cot^2 x$". This is done by using $\cos^2 x = 1 - \sin^2 x$, or, $\cot^2 x \cos^2 x = \cot^2 x - \sin^2 x$.

From this, $\cot^2 x - \cos^2 x = \cot^2 x \cos^2 x$

(3) Show that $\sin^2 x(\tan^2 x - 1) + \cos^2 x(\cot^2 x - 1)$

$$= \frac{(1-2\cos^2 x)^2 \sec^4 x}{\tan^2 x}$$

The integral form on the left and the fractional one on the right suggest that a proper first step is to perform, if possible,

the indicated division. As $\sec^4 x/\tan^2 x = \sec^2 x \csc^2 x$, the right function becomes

$$(1-2\cos^2 x)^2 \sec^2 x \csc^2 x \qquad \text{(a)}$$

Comparison of the left function of (4) with (a) suggests a simplification of the former, namely,

$$\sin^2 x \tan^2 x + \cos^2 x \cot^2 x - (\sin^2 x + \cos^2 x)$$
$$= \sin^2 x \tan^2 x + \cos^2 x \cot^2 x - 1 \qquad \text{(b)}$$

If (a) is identical with (b), it should be possible to make three terms of (a), as (b) is three-termed. This may be done by writing,

$$(1-2\cos^2 x)^2 \sec^2 x \csc^2 x = (\sin^2 x - \cos^2 x)^2 \sec^2 x \csc^2 x, \qquad \text{(c)}$$

a simplification effected by $\sin^2 x + \cos^2 x = 1$, then multiplying out after squaring.

Thus (c) becomes

$$\sec^2 x \csc^2 x (\sin^4 x - 2\sin^2 x \cos^2 x + \cos^4 x)$$
$$= \sec^2 x \sin^2 x - 2 + \csc^2 x \cos^2 x \qquad \text{(d)}$$

Comparison of (d) with (b) shows that to identify the two it should be sufficient to express "$\sec^2 x$" and "$\csc^2 x$" in terms of the tangent and cotangent, respectively, and then insert into (d). Thus (d) becomes

$$\sin^2 x \tan^2 x + \sin^2 x - 2 + \cos^2 x + \cos^2 x \cot^2 x \qquad \text{(e)}$$

Using $\sin^2 x + \cos^2 x + \cos^2 x = 1$, (e) becomes (b)

$$\text{(4) Show: } \frac{\sin^4 x + \cos^4 x}{\sin^2 x \cos^2 x} = \sec^2 x + \csc^2 x$$

Since the left function is expressed in terms of the sine and the cosine only, a first proper step is to express the right function in same terms.

$$\text{Thus, } \sec^2 x + \cos^2 x = 1/\cos^2 x + 1/\sin^2 x$$
$$= 1/\sin^2 x \cos^2 x \qquad \text{(a)}$$

Comparing (a) with the left member of (4) suggests changing the form of the latter to

$$\frac{2\sin^2 x \cos^2 x + \sin^4 x + \cos^4 x}{\sin^2 x \cos^2 x} \qquad \text{(b)}$$

Again (a) and (b) can be identical only if their numerators are identical since their denominators are so.

The form of the numerator of (b) suggests in turn

$$(\sin^2 x + \cos^2 x)^2 = 2\sin^2 x \cos^2 x + \sin^4 x + \cos^4 x$$

or, $\quad 1 = 2\sin^2 x \cos^2 x + \sin^4 x + \cos^4 x$

$\therefore \quad (a) = (b)$

and, (4) is an identity.

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To many experienced and (possibly?) cynical teachers all this may apear trivial or of little consequence. But any practice which aims to habituate the young mind to scientific methods of approach to a problem task is **not** trivial, **not** of little consequence.

*—S.T.S.*

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## MODES OF STUDYING GEOMETRY*

DORA M. FORNO
New Orleans Normal School

"Many teachers have taught subjects, but not how to study subjects. The latter is more important." Thus declared Strayer and Norsworthy years ago and this comment expresses the judgment of many well-known educators of today.

The study period is as important a factor in the student's experience as the teaching period and effective modes of studying geometry are dependent upon effective methods of teaching geometry. The teaching period should not be simply a recitation period but should prepare the student for the subsequent study period. It is a well-known fact that real improvement in study is possible only when the meaning, importance and conditions of study are appreciated. Bagley has said, "To teach a child to study effectively is to do the most valuable thing that could be done to help him adjust himself to an environment of modern civilized life into which he may be thrown."

Independent self direction in the study of geometry, as well as in other branches of learning, should be the goal toward which we are striving, but that goal cannot be reached by one leap and bound.

The colleges and universities are demanding more and more independent work from students and many are failing to reach standards which have been set up because of the lack of power to do independent self directed work. This lack of power is not due wholly to lack of mental ability but to lack of training in concentration and sustained effort. Independent self-directed

work is dependent upon knowing what to do, upon having a motive sufficiently strong to arouse effort to do the work, upon having acquired effective and economical methods of attacking problems, and lastly, upon having the ability of sustained effort and attention to carry on until desired results are attained.

The cause of the student's failure in the first requisite is often traceable to imperfect preparation of the subject matter to be studied due to poor methods of presentation and frequently to too difficult or hurried assignments. Outside intersts are often more appealing than school subjects and as interest and effort are so closely allied, it is often difficult to get a pupil in the proper frame of mind to arouse interest and effort in a chosen task. A problem makes little appeal to the interest of the pupil unless it is a real problem of his own or one that he can be enlisted to make his own. The problem itself may be sufficiently appealing to arouse the active interest of the pupil, but if not, other active interests can be appealed to in order that effort may be maintained. The powers of sustained effort and attention are habits acquired over long periods of time, they are bonds that have been built up through insistence or sticking at a job until it is completed satisfactorily. It is this type of training that should be insisted on from the earliest years of a child's training.

Effective and economical methods of study are as important to the individual to prevent waste of time and energy and reduce cost to a minimum. The ratio of the recitation to the study period varies greatly. McMurry in "How to Study" expresses the belief that one-fourth of the school time should be devoted to study. The problem that is all important is not how much time is to be devoted to study but how that study is carried. It is the mode of studying geometry that we are interested in, and the mode is determined by the type of lesson to be studied, and the necessary equipment and conditions for carrying on individual work.

I wish that every teacher would adopt the slogan: "Geometry understood, not memorized." A large part of high school geometry has degenerated into memorization of theorems and proofs. If one were to examine carefully into the work of the average high school students, the results would convince you that such is the case, but teachers and text-book writers are working assiduously to correct this practice and to free the

pupil from this wholesale memorizing that is prevalent. It must not be inferred that I believe memorizing has no part or only a small part in the study of geometry, for that is not so, but memorizing must be preceded by thoughtful understanding and appreciation of the meaning and use of terms, axioms, theorems, postulates, etc. The recall of facts quickly and surely is one of the aims of education and, to insure this, facts must be studied in association with other facts, hence it is associative memory that we should foster.

All deliberate activities depend upon thought and thought needs safe-guarding and training if the power to reason, to make good judgments, is to be developed, for reasoning, as stated by Thorndike, is "essentially the organization and control of habits of thought." John Locke expressed the great need of safeguarding and training when he said, "It is therefore of highest concernment that great care should be taken of the understanding to conduct it right in the search for knowledge and in the judgments it makes."

The power to think effectively depends upon a number of things, chief of which is native abilities, but whatever the degree of native abilities may be, education has accepted as its task the job of training the individual to the maximum degree of reasoning of which he is capable. It is a big job and educators are often bewildered in the effort to accomplish the task. If the much advocated pedagogical maxim, "One learns by doing" were put more generally into practice, results would be gotten more easily. We cannot make mathematics easy but we can make it easier by the methods of instruction and modes of studying the subject.

No one mode of studying geometry should be advocated as the best mode of getting results any more than one method of teaching should be recommended as the best, for the individuality of the group, as well as the personality of the teacher and the subject matter to be taught, determine largely the mode of studying any topic that best results may be accomplished.

I shall discuss some of the significant modes of studying geometry and illustrate each to show how teachers and textbook writers are using these modes effectively. It is generally conceded that beginners in a new field of learning should not be bemuddled by a multitude of rules and definitions, but that these

rules and definitions should be derived by the students as generalizations of their own after they have had a sufficient number of experiences with concrete materials to bring about the formation of the desired conclusions.

One of the typical modes of studying geometry, especially in the work of observational, or experimental geometry is the "genetic" mode, or the supervised study, wherein teacher and pupil work together making observations, constructions and calculations, the teacher directing the work by skillful questioning and the pupil asking questions for information or direction. For example, I shall use an illustration for sight work in the determination of the equality of triangles. If in Fig. I. $\angle 1=\angle 2$ and $\angle 3=\angle 4$, show that $\triangle ABC=\triangle ADC$.

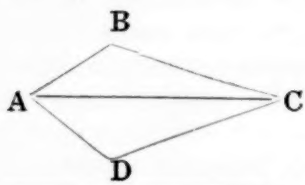



Fig. I.

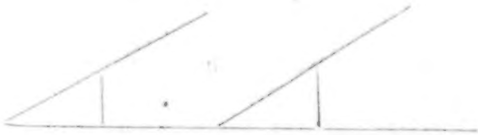
The teacher would suggest that AC is equal to itself. Questioning as to how to show the equality of the two triangles would determination of the equality of triangles. If in Fig. I. $\angle BAC=\angle DAC$ and $\angle BCA=\angle DCA$, show that $\triangle ABC=\triangle ADC$.

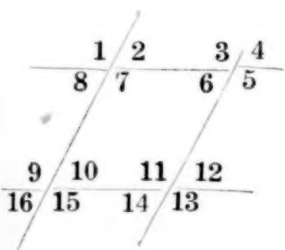
Observation shows that these two triangles have two angles and the included side equal, hence a tentative conclusion is reached and this fact formulated as a proposition.

For a long time it was thought that there was no connection between teaching observational or experimental geometry and demonstrative geometry, but of late years both teachers and text-book writers have shown that best results are attained when geometrical principles are derived inductively through experiences in many situations and after the need for a more logical demonstration has been aroused to present a rigid proof of the theorem. From the work in experimental geometry, the pupil will show some knowledge of geometric figures, will have familiarity with certain phases of constructional work and through problems in area will have had work in computation with geometrical data. One text-book writer uses developmental exercises, sometimes without figures and sometimes with figures, and points

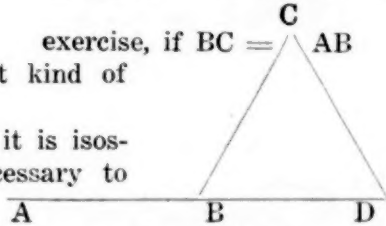
out that figures are necessary if you are to interpret the meaning of your conclusion and carry it further. I will illustrate this transition from the informal to the informal demonstrations by the following problems and simple exercises. One interesting problem showing how by the use of a draughtsman's triangle parallel lines may be drawn can be used as the approach to prove the theorem: Two lines cut by a transversal are parallel if the corresponding angles are equal.

Fig. II.






Another is: Two railway tracks cross as indicated in the figure. What pairs of angles are equal? What pairs of angles are supplementary? This can be used as the approach to the formal proofs of the theorem. If two parallel lines are cut by a transversal the corresponding angles are equal. In the following exercise, if BC = AB and DC = AB, what kind of triangle is BCD? How do you know it is isosceles? Was it necessary to look at the figure to prove BC = DC? Did you look at the figure for any other purpose?




Exercises of this type should be proved very informally at first without regard to mathematical language, but should finally be proved in set form. The following is suggested.

Hypothesis: BC = AB, and DC = AB
Conclusion: $\triangle$BCD is isosceles.

| Proof: | |
|---|---|
| 1. BC = AB | 1. Hypothesis |
| 2. DC = AB | 2. Hypothesis |
| 3. Therefore BC = DC | 3. Things equal to same things are equal to each other. |
| 4. $\triangle$BCD is isosceles. | 4. It has two equal sides. |

A still more formal demonstration may be developed with the final mark of approval stamped at the end, the historical Q. E. D.

It is most important that pupils get fixed in mind the logical procedure for a formal demonstration.

1. Make an accurate construction according to the conditions of the proposition. (A free hand sketch carelessly drawn is a fruitless waste of time.)
2. Construct a general figure.
3. State clearly what is given or constructed according to terms of figure.
4. State clearly what is to be proved.
5. Build up a direct proof.
6. Or try the indirect method.

The "genetic" mode of studying geometry naturally precedes the "individual" mode, for in the "individual" mode the pupil must do independent work along the lines of work previously developed. The individual mode of studying will be determined by the type of lesson to be studied, the degree of preparation that has been received for this study, and the material to carry on the study. In the individual mode the pupil definitely must see the reason or purpose for each step in the development. The pupil must recognize his strength as well as his weak points. It is most important that pupils should be trained to estimate their own abilities and test their own knowledge of subject matter and power of reasoning. Special tests can be devised by the teacher for this purpose or standard tests can be used; for example, the Minnick Geometry Tests.

There is a period in the experience of every individual which is termed by psychologists the "plateau". This is a period of discouragement which is characterized by loss of interest and it is a period to be guarded against if possible or when it does appear definite plans should be made to shorten it. More effort must be made to increase interest, changes should be made in working hours, more time given to reviews, and improvement made in conducting the work, all of which call for skill on the part of the teacher to plan and carry out such a program. The "laboratory" mode of studying geometry has a large part in increasing the interest of the pupil. In this mode the principle that "one learns by doing" has found expression. Equipped with ruler, compass, draughtsman's triangle and protractor, the pupil studies rela-

tions at first hand. The dominant principle in this mode is gaining knowledge through experimentation and not merely accepting the statements of instructors or text-book writers of the ultimate source of information. The pupil learns to evaluate his own work and develops a right attitude towards work, so that he will not waste time in trying to excuse or explain away difficulties but gets to work to improve as much as possible.

*Read before the L. T. A. Mathematics Section Meeting at Alexandria, La.

## PROPOSED FOR DISCUSSION

By PRESIDENT HARRY C. BARBER
National Council of Teachers of Mathematics, Exeter, N. H.

I am going to propose a subject for discussion in the News Letter. It has to do with the conflicting claims, in the algebra class, of teaching for skill and teaching for mental stimulus and the development of intellectual curiosity.

I wish that we might have during the next year or two many contributions on the subject. There must be many readers who have learned by experience how to balance these conflicting claims. Won't they tell us?

Prof. Alfred North Whitehead says "necessary technical skill can only be acquired by a training which is apt to damage those energies of mind which should direct the technical skill."

The question is, how can we prevent the damage?

This quotation is from The Aims of Education, Macmillan. The earlier chapters of this book make stimulating reading for any thoughtful teacher.

## THE NEEDS OF COLLEGE FRESHMEN IN MATHEMATICS

By ERNEST SHIRLEY
L. P. I., Ruston, La.
(Read at the L. T. A. Meeting, Alexandria, La.)

The following information is the result of four years' teaching mathematics at Louisiana Polytechnic Institute.

It is evident that in the short time given me I shall be un-

able to cover the entire field. However, I will endeavor to give some of the outstanding weaknesses of the average student.

In my opinion there is no other subject in the college curriculum more universally used in the various fields of work than mathematics. However, from personal observation I find that there is a large percentage of students who dread and dislike mathematics.

The needs of college freshmen may be divided into the following:

1st. A proper understanding of the basic principles of the subject.

We find students coming to college who are deficient even in the four fundamental operations. The average student is able to work simple problems in ordinary number symbols but when the letter notation is used, he becomes confused. He fails to see that "AxB=AB", though he is able to multiply 6x3. This shows that he is unable to transfer his knowledge of arithmetic to algebra. This bridge from arithmetic to algebra is the beginning of most of his troubles. There are however even in arithmetic certain operations in which a great number of students are deficient. For instance the extraction of a square root, adding fractions, simplifying fractions and operations that involve decimals.

2nd. General weaknesses in algebra.

(a) Work involving the removal of parentheses. Such an expression as $-(X-4)$ is usually called $-X-4$. or,

$$\frac{-2X+3,}{4} \overset{?}{=} \frac{-X+3}{2}$$

(b) The solution of simple equations such as $2X+8=0$. Some students work it: $2X=-8$; $X=-8-2$, $=-10$. A very frequent use of $0/2=2$, is found. There are some students who fail to see that if $16=4X$, then $4X=16$. Surely the student should know that if "$AX+B=O$", then "$X=-B/A$."

(c) Factoring:

Teachers are widely separated on the value of factoring algebraic quantities. However, there is a general need for all the factors of a number. Some students given the factors of 24,

as 2, 3, 4, 6, 8, and 12 when as a matter of fact they are $\pm 1$, $\pm 2$, $\pm 3$, $\pm 4$, $\pm 6$, $\pm 8$, $\pm 12$, $\pm 24$.

(d) Simple fractional equations:

Many students are unable to solve an exercise of this kind: $2/X-1=3/X+2$. He gets it in the form

$2(X+2)/(X-1)(X+2)=3(X-1)/(X+2)(X-1)$, but he does not know what to do next. It is surprising to see how many students do not know when or when not to discard the denominator.

(e) The use of radicals and exponents.

This is one subject that very few students get. They fail to see the relation between radicals and exponents, and some do not know the difference between the two. Less than 25% of them know the fundamental laws of exponents. Common errors are $\sqrt{2}\times\sqrt{3}=\sqrt{5}$; $\sqrt{2}$ multiplied by $\sqrt[3]{7}=\sqrt{14}$. Very few students are able to rationalize the denominator in such as: $1/\sqrt{2}$, $1/\sqrt[3]{2}$. This type is seldom worked: $1/\sqrt{2}\times\sqrt{3}$.

Just to show how little students understand negative and zero exponents, a friend of mine has given this on final examination to thirty-one different classes and has never had a student to work it: "Remove zero and negative exponents and simplify:

$X^{-1}+Y^{-2}/C^{-3}+A^{0}$"

Numerous students cannot solve for X in $1/X=2/3$; or $X^{1/3}=2$.

(f) Quadratic equations:

The student should be able to handle these equations by the three well known methods. Very few can solve by any method. A few can solve by one method. In the use of the formula for the roots of a quadratic equation large numbers of students are unable to assign the proper coefficients. Even those who are able to make the proper substitutions sometimes cannot complete the work.

3rd. Plane Geometry.

We find that the majority of students are able to learn geometry, but the methods used by most students are based upon memory work instead of reasoning. If the theorems could be used as references and the work applied to the exercises the student in my estimation would profit more from geometry.

4th. Lack of examination experience.

In some high schools the student's grades are taken from his daily recitations or papers, and, as we know, many times his work has been done partially or wholly by a classmate. At the close of the term, the student is exempted from examinations. Even though this student has done his own work the system used by all the colleges and universities is the examination system. When the student comes to college, he is unable to assimilate his facts properly for examination. It seems that instead of doing the student a favor he has actually been handicapped.

5th. There is one thing for which none of us are responsible, but which should be remedied. Students are allowed to graduate from high school with one unit of algebra and the colleges require one and one-half units. We have at present some fifty or sixty students enrolled at Tech who are paying a special outside tutor \$5 each for classes to make up this deficiency. This discrepancy which puts the student one quarter behind in his work, and costs him an unnecessary amount of money, might have been remedied in high school. I think this point should receive your immediate attention.

## ON THE INVERSE CIRCULAR FUNCTIONS

By H. L. SMITH
Louisiana State University

The writer has recently had occasion to examine the treatment of inverse circular functions in the college text books. He did not find a single treatment which seemed to him perfect. One was nearly so, but an unfortunate definition of the "principal value" of the inverse cotangent marred even that one.

The trouble is that in the larger sense the inverse circular functions are multiply valued. Since it is important that every mathematical symbol be uniquely defined, nearly all writers define what they call "principal values" of the inverse functions. But in doing this different writers contradict each other and often a writer even contradicts himself.

Some light is thrown on the situation by the consideration

of the function $\sqrt{x}$. In complex function theory this function is necessarily double-valued initially and a two-sheeted Riemann surface is introduced to render it single-valued. But in real function theory the situation is more simple; the domain of definition is restricted to non-negative values of $x$ and $\sqrt{x}$ is defined to be the *POSITIVE* square root of $x$ for such values of $x$. If the negative square root is required it is denoted by $-\sqrt{x}$; if both roots are required they are denoted by $+\sqrt{x}$. This point of view is now almost universal among writers on real function theory, and its advantages in flexibility and precision are obvious.

It is the purpose of this note to show exactly how to treat the inverse circular functions for a real argument in precise analogy with the now current method of treating $\sqrt{x}$ as stated above. The idea of "principal value" is rejected altogether; instead the functions are given unambiguous definitions from the start. It is shown that only gain in precision and flexibility results. It is shown finally that the treatment forms a most satisfactory basis for the introduction of these functions into the calculus.

1. DEFINITIONS OF THE INVERSE FUNCTIONS. It is easily shown that each of the equations.

$$\sin y=x \ (|x|\leq 1), \quad \tan y=x, \quad \csc y=x \ (|x|\geqq 1)$$

has one, and only one, solution which satisfies the inequality

$$-\tfrac{1}{2}\pi\leqq y\leqq\tfrac{1}{2}\pi.$$

These solutions are denoted by arc sin $x$, arc tan $x$, arc csc $x$, respectively. Moreover each of the equations

$$\cos y=x \ (|x|\leqq 1), \quad \text{ctn}\ y=x, \quad \sec y=x \ (|x|\geqq 1)$$

has one, and only one, solution for $y$ which satisfies the inequality $0\leqq y\leqq\pi$. These solutions are denoted by arc cos $x$, arc ctn $x$, arc sec $x$, respectively.

2. RELATIONS BETWEEN THE INVERSE FUNCTIONS. The formulas sin(arc sin $x$)$=x$, cos (arc cos $x$)$=x$, etc. obviously hold. Moreover the formula

$$(1) \quad \cos(\text{arc}\sin x)=\sqrt{1-x^2} \qquad (|x|\leqq 1)$$

holds. For since sin (arc sin $x$)$=x$, it follows that cos (arc sin $x$) is either $\sqrt{1-x^2}$ or $-\sqrt{1-x^2}$. But it is not the latter since arc sin $x$ is by definition between $-\pi/2$ and $\pi/2$ and hence its cosine is non-negative. Hence the formula is proved. In a similar fashion we may prove the formula.

(2) $\sin(\text{arc cos } x)=\sqrt{1-x^2}$ $(|x|\leq 1)$,
and the other formulas for the direct functions of the inverse functions of $x$.

Also the formula

(3) $\text{arc sin x}+\text{arc cos } x=\frac{1}{2}\pi$ $(|x|\leq 1)$
holds. For by (1), (2),

$$\sin(\text{arc sin } x+\text{arc cos } x)=x^2+(1-x^2)=1.$$

Morevover by definitions of arc sin $x$ and arc cos $x$, arc sin $x$+arc cos $x$ lies between $-\frac{1}{2}\pi$ and $3/2\pi$. Hence as sin $\frac{1}{2}\pi$ is the only value of $y$ between $-\frac{1}{2}\pi$ and $3/2\pi$ for which sin $y$ $=1$, the formula is proved. In similar fashion the formulas

(4) $\text{arc tan } x+\text{arc ctn } x=\frac{1}{2}\pi$

(5) $\text{arc sec } x+\text{arc csc } x=\frac{1}{2}\pi$ $(|x|\geq 1)$

may be proved.

In the same way the formula

(6) $\text{arc tan } x-\text{arc tan } y=\text{arc tan }[(x-y)/(1+xy)]$

may be proved to hold, subject to the restriction that $x$ and $y$ are such that arc tan $x$—arc tan $y$ lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ and that $xy$ shall not equal $-1$. Other formulas similar to (6) may also be proved.

The formulas

(7) $\text{arc sin } x=\text{ arc tan }\dfrac{x}{\sqrt{1-x^2}}$,

(8) $\text{arc sec } x=\text{ arc cos }\dfrac{1}{x}$

will be needed later.

3. SOLUTION OF EQUATIONS. It is easily shown that the complete solution of the equation

$$\sin x=0$$

is given by

$$x=n\pi,$$

where $n$ denotes any positive or negative or zero. Also the complete solution of

$$\cos x=0$$

is given by

$$x=\pi/2+n\pi.$$

Now let it be required to solve

(9) $\sin \text{x}=\sin c$,

or $\sin x-\sin c=0$.

On factoring the left member of this equation, it becomes

$$2\cos\frac{x+c}{2}\sin\frac{x-c}{2}=0.$$

which is equivalent to

$$(10)\quad \cos\frac{x+c}{2}=0,\quad \sin\frac{x-c}{2}=0.$$

The first equation of (10) gives

$$\frac{x+c}{2}=\frac{\pi}{2}+n\pi$$

or $\qquad x=(2n+1)\pi-c$

or (11) $x=(2n+1)\pi+(-1)^{2n\times 1}\,c$

The second equation of (10) gives

$$\frac{x-c}{2}=n\pi$$

or $\qquad x=2n\pi+c$

or (12) $x=2n+(-1)^{2n}c$

But equations (11), (12) may be written as the single equation

(13) $\qquad x=n\pi+(-1)^{n}c,$

which is the complete solution of (9).

The equation

(14) $\qquad \sin x=a$

is now easily solved. For it may be written

$$\sin x=\sin(\arcsin a).$$

Hence by (13) the general solution of (14) is given by

(15) $\qquad x=n\pi+(-1)^{n}\arcsin a.$

In similar fashion it may be shown that the general solution of

(16) $\qquad \cos x=a$

is given by

(17) $\qquad x=2n\pi\pm\text{arc}\cos a;$

that the general solution of

(18) $\qquad \tan x=a$

is given by

(19) $\qquad x=n\pi+\arctan a;$

that the general solution of

(20) $\operatorname{ctn} x = a$

is given by

(21) $x = n\pi + \operatorname{arc\,ctn} a$;

that the general solution of

(22) $\sec x = a$

is given by

(23) $x = 2n\pi \pm \operatorname{arc\,sec} a$,

and finally that the general solution of

(24) $\csc x = a$

is given by

(25) $x = n\pi + (-1)^n \operatorname{arc\,csc} a$.

Equations (15), (17), (19), (21), (23), (25) may be regarded as giving the complete expressions for the inverse circular functions, just as $\sqrt{x}$ is the complete expression for the square root function.

4. DIFFERENTIATION OF THE INVERSE CIRCULAR FUNCTIONS. Let it be required to find a formula for D arctan $x$, derivative with respect to $x$ of arctan $x$. This is, by definition, the limit as $h$ approaches zero of $[\arctan(x+h) - \arctan x]/h$. But, by (6), this is equal to $[\arctan(hf)]/h$ where $f = 1/[1+x(x+h)]$. Hence, since

$$[\arctan(hf)]/h = [\arctan(hf)/hf]f,$$

and the limit of $f$ as $h$ approaches zero is $1/(1+x^2)$, so that the limit of $hf$ is zero, it follows that the limit of $[\arctan(x+h) - \arctan x]/h$ is $1/(1+x^2)$. Thus the formula

(26) $D \arctan x = 1/(1+x^2)$

has been proved.

From formulas (7), (3), (4), (8), (5), formulas may now successively be found for the derivatives of arcsin $x$, arc cos $x$, arc csc $x$, respectively. The formulas in the cases of the two latter are

(27) $D \operatorname{arc\,sec} x = 1/|x|\sqrt{x^2-1}$,

(28) $D \operatorname{arc\,csc} x = -1/|x|\sqrt{x^2-1}$,

which agree with the incorrect formulas usually given only for positive values of $x$.

It is to be noted that the above procedure not only obtains the formulas for these derivatives without sacrifice of simplicity but also proves their **existence** and with a gain in directness

The difficult matter of implicit functions is avoided altogether.

5. CONTINUITY OF THE INVERSE FUNCTION. Since continuity is necessary for the existence of a derivative, and since each of the inverse functions has been shown to have a derivative at every point **interior to** its domain of definition, it follows that each of those functions is continuous at all such points.

6. CONCLUSION. Much has been claimed for the treatment of the inverse circular function sketched above. The writer hopes that every reader who teaches trigonometry, and especially those who also teach calculus, will check these claims. It is believed that the editor of the NEWS LETTER will be glad to offer space for discussion of any points which may be raised.

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## PROBLEMS

Proposed by W. PAUL WEBBER

1. A ladder 20 ft. long stands vertically against a wall. A cat starts up the ladder and at the same time the foot of the ladder is pulled away from the wall at the same rate that the cat climbs. What will be the greatest elevation from the ground the cat can attain while the ladder is brought to a horizontal position on the ground?
2. A circular garden 10 ft. in diameter is fenced. A horse is tied to the fence on the outside of the garden by a rope 100 ft. long. Over how much ground can the horse graze?
3. If squares are constructed on the four sides of a parallelogram exterior to it and in its plane, show that the quadrilateral determined by connecting the centres of these squares is also a square.

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## SOLUTIONS OF PROBLEMS

Proposed by W. PAUL WEBBER

A horse is tied to the outside of a circular fence 100 ft. in diameter by a rope 100 ft. long. Over how much ground can the horse graze?

**Solved by F. A. Rickey.** We see that the area reached by

the horse is the area of a semi-circle of radius R(length of rope) plus two equal iregular areas generated by the rope R as it wraps around the circle of radius $r$ incidentally forming a bounding curve which is an involute of the circular arc. We denote by $l$ the length of the portion of the rope tangent to the circle at any instant and note that the angle between two positions of $l$ is equal to the angle between two corresponding positions of the radius of the circle. Hence if $v$ is the angle between the radius to the point of attachment of the rope and any other radius, $dv$ will represent the angle between $l$ and $l+dl$.

We now choose as an element of area the area bounded by two positions of $l$, the intercepted arc of the circle and the intercepted portion of the involute. This area differs from $\frac{1}{2}l^2dv$ by a vanishingly small amount (See P. 134, Byerly's Integral Calculus). Thus observing that $v$ varies from 0 to R/$r$ and denoting the irregular area by S, we write $dS = \frac{1}{2}l^2dv$

$$S = \frac{1}{2}\int_0^{R/r} l^2dv \text{ which becomes}$$

$$S = \frac{1}{2}\int_0^{R/r} (R-rv)^2dv \text{ upon replacing } l \text{ by its equal } R-rv.$$

$$\therefore S = \frac{1}{2}\int_0^{R/r} (R^2-2rRv+r^2v^2)dv.$$

$$= \frac{1}{2}\left[R^2v-rRv^2+r^2v^3/3\right]_0^{R/r} = R^3/6r$$

Hence the total area considered is

$$A = 2(R^3/6r)+\pi R^2/2 = R^2(R/3r+\pi/2)$$

Substituting 100 for R and 50 for $r$ we obtain

$$10000(100/150+\pi/2) = 10000(2/3+\pi/2)$$

or about 22,374-2/3 sq. ft.

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### The General Case*

A brief discussion of the general problem of determining the area swept out by a line moving so as to continually touch two given curves will be timely here. Here the restriction found above as to type of curve and tangency of the line to one of them is removed.

Consider two curves $f_1(t)$ and $f_2(t)$ and let $(x_1, y_1)$ and $(x_2, y_2)$ be moving points on the two respective curves, their positions being determined by the parameter $t$. Let $l$ be the moving line joining these points. We wish to find a formula for the area S qenerated by $l$ as it moves between two positions determined by $t=t_0$ and $t=t_1$.

Taking as an element of area the quadrilateral determined by the points $(x_1,y_1),(x_2,y_2)(x_2+dx_2,\ y_2+dy_2)$ and $(x_1+dx_1,\ y_1+dy)$, we note that it may be expressed as the sum of the areas of the two triangles formed by a diagonal of the quadrilateral. These areas may be obtained by the regular method of analytic geometry. Using the fact that the area of a triangle can be expressed in determinate form from the cordinates of its vertices we obtain.

$$dS = \tfrac{1}{2}\begin{vmatrix} x_1+dx_1 & y_1+dy_1 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \tfrac{1}{2}\begin{vmatrix} x_2+dx_2 & y_2+dy_2 & 1 \\ x_1+dx_1 & y_1+dy_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$$

$$= \tfrac{1}{2}[(y_1-y_2)dx_1-(x_1-x_2)dy_1+(y_1-y_2)dx_2-(x_1-x_2)dy_2]$$

$$= \tfrac{1}{2}[(y_1-y_2)(dx_1+dx_2)-(x_1-x_2)\ (dy_1+dy_2)]$$

where products such as $dx\cdot dy_1$ have been discarded.

We therefore have

$$(1)\qquad S=\tfrac{1}{2}\int_{t_0}^{t_1}[(y_1-y_2)(dx_1+dx_2)-(x_1-x_2)(dx_1+dy_2)]$$

By letting $u$ represent the angle $l$ makes with the horizontal and noting that

$$x_2=x_1+l\cos u,\quad dx_2=dx_1+\cos u\,dl-l\sin u\,du$$
$$y_2=y_1+l\sin u,\quad dy_2=dy_1+\sin u\,dl+l\cos u\,du$$

we find that (1) becomes:

$$S=\tfrac{1}{2}\int[l\sin u(2dx_1+\cos u\,dl-l\sin u\,du)$$
$$-l\cos u(2dy_1+\sin u\,dl+l\cos u\,du)]$$

which reduces to

$$(2)\qquad S=\tfrac{1}{2}\int l(\sin u\,dx_1-\cos u\,dy_1)-\tfrac{1}{2}\int_{v_0}^{v_1} l^2du$$

Not only will this formula reduce to that used to determine the S of the original special problem under the conditions found there, but interesting reductions to well known area formulae can be easily obtained.

In the case of the grazing problem considered we find that

$u=\Pi/2+v, \quad du=dv$ and

$x_1=r\cos v, \quad y_1=r\sin v$. Under substitution of these values, (2) becomes

$$S=\int_{v_0}^{v_1} l[-\sin(\Pi/2+v)\sin v-\cos(\Pi/2+v)\cos v]dv-\tfrac{1}{2}\int_{v_0}^{v_1} l^2dv$$

$$=\int_{v_1}^{v_0} l[-\cos v\sin v+\sin v\cos v]dv-\tfrac{1}{2}\int_{v_0}^{v_1} l^2dv$$

$$=-\tfrac{1}{2}\int_{v_0}^{v_1} l^2dv, \text{ as used in the solution.}$$

---

Now consider the area bounded by two ordinates, $x$—axis, and a simple curve, as being generated by a moving ordinate $y$. Let us notice what happens to (2) under these conditions. Here $u=\Pi/2$, $du=0$, $y_1=dy_1=0$, $l=y$, and $x_1=x$, giving the well known formula

$$S=\int y(\sin\Pi/2dx-\cos\Pi/2\cdot 0)-\tfrac{1}{2}\int_{v_0}^{v_1} y^2\cdot 0$$

$$=\int_a^b ydx$$

---

The formula in polar cordinates for the area swept out by a radius vector may be easily obtained. Here

$x_1=y_1=dx_1=dy_1=0$

and (2) becomes

$$S=\int l(\sin u\cdot 0-\cos u\cdot 0)-\tfrac{1}{2}\int_{v_1}^{v_0} l^2du$$

$$=-\tfrac{1}{2}\int_{v_0}^{v_1} l^2du$$

*From suggestions and formulae by H. L. Smith.

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It is important that district leaders remember the following: Every check for $1.00 (subscription) nets but 90c to the Mathematics News Letter unless consolidated with several subscriptions. Several subscriptions should be consolidated in one check.

## EDITORIAL CORRESPONDENCE

The very frank paper of Mr. E. M. Shirley—published in this issue of the News Letter—regarding the shortcomings of the average freshman in elementary algebra makes appropriate the publication of the following letter from Professor John C. Stone to Editor Forno. Though it is undeniably true that algebra is preeminently the field of freshman trouble, it is nevertheless thoroughly gratifying to know that the causes of these difficulties are being subjected to scientific investigation by such competent specialists as Professor Stone and, doubtless, many others.

State Teachers College, Upper Montclair, N. J.,
November 17, 1929.

My dear Miss Forno:

I have yours of the 12th enclosing a copy of "Mathematics News Letter." I've read it through this morning and it should be of value to teachers of the subject. The articles are all full of valuable suggestions.

You ask for a contribution. I'm very busy but I never refuse my friends. I am now making a careful study of "Testing" in algebra. Last Spring I gave a test to 330 H. S. graduates from 100 different high schools, all ranking in the upper half of the class to see what they **did** and **did not** get from this H. S. course in algebra. A "write-up" of this might be very helpful to teachers of First Year Algebra.

I've just given a diagnostic test to 9th grade algebra students who have covered work in evaluating formulas and in the four fundamental processes. These tests showed many types of difficulty that pupils have, and hence many phases of the work that most teachers overlook in teaching the subject.

Next week I'm giving four inventory tests in the fundamental skills in each of the four processes. These tests include every possible variation that pupils meet—about 100 questions in each—in an attempt to see just what difficulties arise in getting a complete control of these four fundamental processes, each of which is composed of a large number of distinct skills.

So I could furnish you a very valuable article on the results of these tests. How much space could I have, and which of the above suggestions make the greatest appeal to you?

I'm trying to write a paper on "Some reasons why we should have a combined course in plane and solid geometry for the tenth grade", for the meting at Atlantic City in February. Can you give me any suggestions?

Very sincerely,

JOHN C. STONE



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